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A Note on the Newton–Cotes Integration Formula

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If $\Phi(x)$ is defined on $[-1, 1]$, let $L_n(\Phi, x)$ denote the Lagrange interpolation polynomial of degree n (or less) which agrees with $\Phi(x)$ at the equidistant nodes $x_{k,n} = -1 + (2k)/n$ ($k = 0, 1, ..., n$). The classical Newton–Cotes integration formula approximates $\int_{-1}^{1} \Phi(x)\,dx$ by $\int_{-1}^{1} L_n(\Phi, x)\,dx$. In this paper we present a very simple example of an analytic function $\Phi(x)$ for which $\lim_{n\to\infty} \int_{-1}^{1} L_n(\Phi, x)\,dx \neq \int_{-1}^{1} \Phi(x)\,dx$.

If $\Phi(x)$ is a real-valued function defined on the interval $[-1, 1]$, the Lagrange interpolation polynomial of degree n (or less) which agrees with $\Phi(x)$ at the equidistant nodes

$$x_{k,n} = x_k = -1 + (2k)/n, \qquad k = 0, 1, 2, ..., n,$$

will be denoted by $L_n(\Phi, x)$. It is well-known (see Davis and Rabinowitz [1]) that

$$\int_{-1}^{1} L_n(\Phi, x)\,dx = \sum_{k=0}^{n} \lambda_{k,n} \Phi(x_{k,n}),$$

and this formula is often called the Newton–Cotes integration formula.

A classical result in numerical integration is that there is a continuous function Φ on $[-1, 1]$ such that

$$\lim_{n\to\infty} \int_{-1}^{1} L_n(\Phi, x)\,dx \neq \int_{-1}^{1} \Phi(x)\,dx.$$

This result follows immediately from each of the independent results of J. Ouspensky [4] (1925), R. O. Kusmin [2] (1931), and G. Pólya [5]

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(1933). (Note that "Ouspensky" is usually spelt as "Uspensky," but not in [4], and that Kusmin's proof is the basis of the proof of this result presented in Natanson [3, pp. 129–136].)

Unlike the proofs of Ouspensky and Kusmin, Pólya's proof is constructive. He shows that

$$\lim_{n \to \infty} \int_{-1}^{1} L_n(g, x)\, dx \neq \int_{-1}^{1} g(x)\, dx,$$

where

$$g(x) = -\sum_{k=4}^{\infty} a^{k!} \sin(k!\, \pi(x+1)/2) \sec(\pi(x+1)/2)$$

and $1/2 < a < 1$. Pólya's function not only is continuous on $[-1, 1]$, but is also analytic on this interval. However, it is a very complicated example.

The aim of this paper is to present a much simpler example based on a famous result of C. Runge [6] (1901) which states that if

$$h(x) = (1 + 25x^2)^{-1}, \qquad -1 \leqslant x \leqslant 1,$$

then the sequence $\{L_n(h, x) : n = 1, 2, 3, ...\}$ does not converge to $h(x)$ uniformly on $[-1, 1]$. An immediate consequence of the theorem below is that

$$\lim_{n \to \infty} \int_{-1}^{1} L_n(h, x)\, dx \neq \int_{-1}^{1} h(x)\, dx,$$

and hence we have a very simple function $h(x)$ which is analytic on $[-1, 1]$ and for which the Newton–Cotes integration formula is divergent. Our main result is as follows.

THEOREM. *Suppose $N > 0$, and define*

$$f_N(x) = \frac{1}{1 + N^2 x^2}.$$

Then, as $n \to \infty$,

$$\log\left\{(-1)^{1+[n/2]} \int_{-1}^{1} (L_n(f_N, x) - f_N(x))\, dx\right\}$$
$$= n\left[\log\left(\frac{2N}{\sqrt{N^2+1}}\right) + \frac{\tan^{-1}(N^{-1})}{N} - \frac{\pi}{2N}\right] - \frac{3}{2}\log n - 2\log\log n$$
$$+ \log\left(\frac{8N\sqrt{2\pi}}{(N^2+1)^{3/2}}\right) + \left(\frac{1+(-1)^n}{2}\right)\log N + O((\log n)^{-1}). \qquad (1)$$

Hence the statement

$$\lim_{n\to\infty}\int_{-1}^{1} L_n(f_N, x)\,dx = \int_{-1}^{1} f_N(x)\,dx$$

holds true if and only if $N \leqslant N_0$, *where* N_0 (≈ 1.9029) *is the single positive root of the equation*

$$\log\left(\frac{2N}{\sqrt{N^2+1}}\right) + \frac{\tan^{-1}(N^{-1})}{N} - \frac{\pi}{2N} = 0.$$

Before establishing the theorem, we need to obtain a convenient representation for the error in approximating $\int_{-1}^{1} f_N(x)\,dx$ by $\int_{-1}^{1} L_n(f_N, x)\,dx$.

LEMMA. *Define*

$$G_n(t) = \pi^{-1}\sin(\pi t)\,\Gamma(n+1-t)\,\Gamma(1+t).$$

Then

$$\int_{-1}^{1} (L_n(f_N, x) - f_N(x))\,dx$$

$$= \frac{2\pi}{N\,|\Gamma(n/2+1+in/2N)|^2}$$

$$\times \operatorname{Im}\left[\operatorname{cosec}\left(\frac{n\pi}{2} + \frac{in\pi}{2N}\right)\int_0^{n/2} \frac{G_n(t)}{n/2+in/2N-t}\,dt\right]. \tag{2}$$

Proof. From Lagrange's formula for interpolation polynomials we have

$$L_n(f_N, x)$$

$$= \sum_{k=0}^{n} (-1)^k f_N(x_k)$$

$$\times \frac{(n-n(1+x)/2)(n-1-n(1+x)/2)\cdots(1-n(1+x)/2)(-n(1+x)/2)}{(k-n(1+x)/2)\,k!(n-k)!}$$

$$= G_n\left(\frac{n(1+x)}{2}\right)\sum_{k=0}^{n} \frac{(-1)^{k-1} f_N(x_k)}{(k-n(1+x)/2)\,k!(n-k)!}. \tag{3}$$

Also,

$$f_N(x_k) = \frac{1}{1+N^2x_k^2} = \operatorname{Re}\left(\frac{1}{1+iNx_k}\right)$$

$$= \frac{n}{2N}\operatorname{Im}\left(\frac{1}{k-n/2-in/2N}\right). \tag{4}$$

By using (3) and (4), together with the fact that $L_n(f_N, x)$ is even, we obtain

$$\begin{aligned}
\int_{-1}^{1} & L_n(f_N, x)\, dx \\
&= 2\int_{-1}^{0} L_n(f_N, x)\, dx \\
&= \frac{n}{N}\int_{-1}^{0} G_n\left(\frac{n(1+x)}{2}\right) \\
&\quad \times \operatorname{Im}\left[\sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k-n(1+x)/2)(k-n/2-in/2N)\, k!(n-k)!}\right] dx \\
&= \frac{2}{N}\int_{0}^{n/2} G_n(t) \operatorname{Im}\left[\sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k-t)(k-n/2-in/2N)\, k!(n-k)!}\right] dt. \qquad (5)
\end{aligned}$$

We will need the notation

$$(a)_k = \begin{cases} 1, & \text{if } k=0, \\ a(a+1)(a+2)\cdots(a+k-1), & \text{if } k=1, 2, 3, \ldots, \end{cases}$$

and the hypergeometric function is defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\, (b)_k}{(c)_k} \frac{z^k}{k!}.$$

Then, if t is not an integer, we can write

$$\begin{aligned}
\sum_{k=0}^{n} & \frac{(-1)^{k+1}}{(k-t)(k-n/2-in/2N)\, k!(n-k)!} \\
&= \frac{1}{t-n/2-in/2N}\left[\sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k-t)\, k!(n-k)!} \right.\\
&\quad \left. - \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k-n/2-in/2N)\, k!(n-k)!}\right] \\
&= \frac{1}{n!(t-n/2-in/2N)}\left[\frac{1}{t}\sum_{k=0}^{n} \frac{(-n)_k\, (-t)_k}{(-t+1)_k\, k!} \right.\\
&\quad \left. - \frac{1}{(n/2+in/2N)}\sum_{k=0}^{n} \frac{(-n)_k\, (-n/2-in/2N)_k}{(-n/2-in/2N+1)_k\, k!}\right] \\
&= \frac{1}{n!(t-n/2-in/2N)}\left[\frac{1}{t} F(-n, -t; -t+1; 1) \right.\\
&\quad \left. - \frac{2N}{n(N+i)} F\left(-n, -\frac{n}{2}-\frac{in}{2N}; -\frac{n}{2}-\frac{in}{2N}+1; 1\right)\right].
\end{aligned}$$

From the well-known result

$$F(a, b; c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$

which is valid if c is not a negative integer or zero, and $\operatorname{Re}(c-a-b) > 0$, we deduce that

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k-t)(k-n/2-in/2N)\,k!(n-k)!}$$
$$= \frac{1}{t-n/2-in/2N}\left[\frac{1}{t}\frac{\Gamma(1-t)}{\Gamma(n+1-t)} - \frac{2N}{n(N+i)}\frac{\Gamma(-n/2-in/2N+1)}{\Gamma(n/2-in/2N+1)}\right]$$
$$= \frac{1}{t-n/2-in/2N}$$
$$\times\left[\frac{1}{G_n(t)} - \frac{2N}{n(N+i)}\frac{\pi}{\Gamma(n/2+in/2N)\,\Gamma(n/2+1-in/2N)\sin(n\pi/2+in\pi/2N)}\right]$$
$$= \frac{1}{t-n/2-in/2N}\left[\frac{1}{G_n(t)} - \frac{\pi\operatorname{cosec}(n\pi/2+in\pi/2N)}{|\Gamma(n/2+1+in/2N)|^2}\right]. \tag{6}$$

Substituting (6) into (5) yields

$$\int_{-1}^{1} L_n(f_N, x)\,dx = \frac{2}{N}\operatorname{Im}\int_0^{n/2}\frac{dt}{t-n/2-in/2N}$$
$$+\frac{2\pi}{N\,|\Gamma(n/2+1+in/2N)|^2}$$
$$\times\operatorname{Im}\left[\operatorname{cosec}\left(\frac{n\pi}{2}+\frac{in\pi}{2N}\right)\int_0^{n/2}\frac{G_n(t)}{n/2+in/2N-t}\,dt\right].$$

The required statement (2) then follows from the observation that

$$\frac{2}{N}\operatorname{Im}\int_0^{n/2}\frac{dt}{t-n/2-in/2N} = 2\,\frac{\tan^{-1}N}{N} = \int_{-1}^{1} f_N(x)\,dx.$$

Proof of the Theorem. We begin by using an approach due to Kusmin [2] to investigate the asymptotic behaviour of the integral

$$\int_0^{n/2}\frac{G_n(t)}{n/2+in/2N-t}\,dt$$

that appears in the right-hand side of (2). Define

$$\alpha = \frac{\log\log n + \log\log\log n}{\log n},$$

and write

$$\int_0^{n/2} \frac{G_n(t)}{n/2+in/2N-t}\,dt = \int_0^{\alpha} + \int_{\alpha}^{2} + \int_2^{n/2} \frac{G_n(t)}{n/2+in/2N-t}\,dt$$

$$= I_1 + I_2 + I_3. \tag{7}$$

We first consider I_3. If $2 \leqslant t \leqslant n/2$, then

$$\left|\frac{n}{2}+\frac{in}{2N}-t\right|^{-1} \leqslant \frac{2N}{n},$$

and

$$\Gamma(n+1-t)\,\Gamma(1+t) \leqslant \Gamma(n-1)\,\Gamma(2) = (n-2)!.$$

Thus

$$|I_3| \leqslant \left(\frac{n}{2}-2\right)\frac{(n-2)!}{\pi}\frac{2N}{n},$$

and so, as $n \to \infty$,

$$I_3 = O((n-2)!). \tag{8}$$

Next, consider I_2. From the Maclaurin expansion of $\log \Gamma(n+1-t)$, it follows that for $0 \leqslant t \leqslant 2$, we have, as $n \to \infty$,

$$\log \Gamma(n+1-t) = \log \Gamma(n+1) - t \log n + O(n^{-1}),$$

or

$$\Gamma(n+1-t) = n!\, e^{-t \log n}(1+O(n^{-1})). \tag{9}$$

Furthermore, for $0 \leqslant t \leqslant 2$ we have $\Gamma(1+t) \leqslant 2$, $|\sin \pi t| \leqslant \pi t$, and

$$\left(\frac{n}{2}+\frac{in}{2N}-t\right)^{-1} = \frac{2N}{n(N+i)}(1+O(n^{-1})). \tag{10}$$

Thus, as $n \to \infty$,

$$|I_2| = O\left((n-1)! \int_{\alpha}^{\infty} t e^{-t \log n}\,dt\right)$$

$$= O\left((n-1)! \frac{\alpha e^{-\alpha \log n}}{\log n}\right)$$

$$= O\left(\frac{(n-1)!}{(\log n)^3}\right). \tag{11}$$

Finally, consider I_1. Upon using (9) and (10), together with the relations $\Gamma(1+t)=1+O(t)$ and $\sin \pi t=\pi t+O(t^3)$ for $0\leqslant t\leqslant \alpha$, we find

$$I_1=\frac{2N(n-1)!}{N+i}\left(\int_0^{\alpha} te^{-t\log n}\,dt+O\left(\int_0^{\infty} t^2e^{-t\log n}\,dt\right)\right)$$
$$=\frac{2N(n-1)!}{(N+i)(\log n)^2}\left(1+O\left(\frac{1}{\log n}\right)\right). \tag{12}$$

Using (8), (11), and (12) in (7), we obtain, as $n\to\infty$,

$$\int_0^{n/2}\frac{G_n(t)}{n/2+in/2N-t}\,dt=\frac{2N(N-i)}{N^2+1}\frac{(n-1)!}{(\log n)^2}\left(1+O\left(\frac{1}{\log n}\right)\right).$$

Now, as $n\to\infty$, we have

$$\operatorname{cosec}\left(\frac{n\pi}{2}+\frac{in\pi}{2N}\right)=-2ie^{in\pi/2}e^{-n\pi/(2N)}(1+O(e^{-n\pi/N})),$$

and so

$$\operatorname{Im}\left[\operatorname{cosec}\left(\frac{n\pi}{2}+\frac{in\pi}{2N}\right)\int_0^{n/2}\frac{G_n(t)}{n/2+in/2N-t}\,dt\right]$$
$$=-\frac{4N}{N^2+1}\frac{(n-1)!}{(\log n)^2}e^{-n\pi/(2N)}\left(1+O\left(\frac{1}{\log n}\right)\right)\operatorname{Re}[e^{in\pi/2}(N-i)]$$
$$=(-1)^{1+[n/2]}\frac{4N}{N^2+1}N^{(1+(-1)^n)/2}\frac{\Gamma(n)}{(\log n)^2}e^{-n\pi/(2N)}\left(1+O\left(\frac{1}{\log n}\right)\right). \tag{13}$$

Substituting (13) in (2), and using the well-known asymptotic expansion

$$\log\Gamma(z)=(z-\tfrac{1}{2})\log z-z+\tfrac{1}{2}\log(2\pi)+O(z^{-1}),$$

as $z\to\infty$, we obtain the required result (1).

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